

Analytic Solitary-wave Solutions for Modified Korteweg - de Vries Equation with t -dependent Coefficients

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We find analytic solitary wave solutions for a modified KdV equation with t -dependent coefficients of the form $u_t - 6\alpha(t)uu_x + \beta(t)u_{xxx} - 6\gamma u^2u_x = 0$. We make use of both the application of the truncated Painlevé expansion and symbolic computation to obtain an auto-Bäcklund transformation. We show that kink-type analytic solitary-wave solutions exist under some constraints on $\alpha(t)$, $\beta(t)$ and γ .

Key words: Variable-coefficient Modified KdV Equation; Truncated Painlevé Expansion; Bäcklund Transformation; Analytic Solitary-wave Solutions.

Korteweg-de Vries (KdV) type equations have been the most important non-linear evolution equations, with numerous applications in physical sciences and engineering. However, the physical situations in which the KdV equation arises tend to be highly idealized, due to the assumption of constant coefficients, e. g., in the propagation of small-amplitude surface waves in fluids of constant depth. However, more realistic physical nonlinear evolution equations contain variable nonlinear or/and dispersive coefficients. An interesting extension of the standard KdV equation is the modified variable coefficients KdV (mvcKdV) equation [1, 2], which models long wave propagation in an inhomogeneous two-layer shallow liquid with a variable nonlinear coefficient $\alpha(t)$ and dispersion coefficient $\beta(t)$,

$$u_t - 6\alpha(t)uu_x + \beta(t)u_{xxx} - 6\gamma u^2u_x = 0, \quad (1)$$

where $u(x, t)$ is proportional to the elevation of the interface between two layers, the evolution variable t has the meaning of the propagation coordinate, x is the coordinate in the reference frame moving with a mean group velocity of the waves. The dependences $\alpha(t)$ and $\beta(t)$ imply that the ratio of the depths of the two layers may depend upon the coordinate, [3]. The mvcKdV equation can also describe the propagation

of nonlinear ion acoustic waves in a plasma containing inhomogeneous negative ions [4]; the coefficient $\alpha(t)$ vanishes at a certain critical density of the negative ions. Malomed and Shrira [3] have found some analytic solitary-wave solutions for the mvcKdV and studied their behavior for the waves passing through the critical point where either $\alpha(t)$ or $\beta(t)$ vanishes or changes its sign. They showed that a primordial soliton passing the critical point transforms into a train of secondary solitons of the opposite polarity. More recently, Talipova et al. [5] and Grimshaw et al. [6, 7] numerically investigated the characteristics of solitary wave transformation in a zone with the sign-changing coefficient $\alpha(t)$ and confirmed the results of [3]. However, exact analytic solitary-wave solutions for the mvcKdV have not been found yet in the previous works [3, 5, 6].

On the other hand, it was shown by Joshi [8] that the mvcKdV equation with $\gamma = 0$ possesses the Painlevé property if and only if its respective coefficient functions satisfy the constraint

$$\beta(t) = -6\alpha(t) \left(-6a_0 \int^t \alpha(s) ds + b_0 \right), \quad (2)$$

where a_0 and b_0 are arbitrary constants. In deriving the constraint, the author used a simple Ansatz function

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for $\phi(x, t) = x + \zeta(t)$ near $\phi(x, t) = 0$, where $\zeta(t)$ is an arbitrary function of t , due to the non-characteristic ($\phi_x \neq 0$) nature of the manifold. This property guarantees that the equation can be mapped to the usual KdV equation which is known to be completely integrable. More recently, by using the truncated Painlevé expansion and symbolic computation methods in [9, 10], Hong and Jung [11] have obtained an auto-Bäcklund transformation and analytic solitonic solutions of (1) for the case of $\gamma = 0$, using a more general Ansatz for $\phi(x, t)$ than that of [8]. In this paper we make use of the methods in [11] to show the existence of an auto-Bäcklund transformation and analytic kink-type solitary-wave solutions with some constraints on the model parameters $\alpha(t)$, $\beta(t)$, and γ .

A non-linear partial differential equation (NPDE) is said to possess the Painlevé property if the solutions of the NPDE are “single valued” about the movable, singularity manifold which is “non-characteristic”. To be more precise, if the singularity manifold is determined by

$$\phi(z_1, z_2, \dots, z_n) = 0, \quad (3)$$

and $u = u(z_1, z_2, \dots, z_n)$ is a solution of the NPD, then it is required that

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j, \quad (4)$$

where $u_0 \neq 0$, $\phi = \phi(z_1, z_2, \dots, z_n)$, $u_j = u_j(z_1, z_2, \dots, z_n)$ are analytic function of (z_j) in a neighborhood of the manifold (4) and α is a negative, rational number. Substitution of (4) into the NPDE determines the allowed values of α and defines the recursion relation for u_j , $j = 0, 1, 2, \dots$. When the Ansatz (4) is correct, the NPDE is said to possess the Painlevé property and is conjectured to be integrable [12, 13, 14].

In order to find solitary-wave solutions of (1), we truncate the Painlevé expansion at the constant-level term in the senses of Hong and Jung [11] and Hong [15],

$$u(x, t) = \phi^{-N}(x, t) \sum_{j=0}^N u_j(x, t) \phi^j(x, t). \quad (5)$$

On balancing the highest-order contributions from the linear term (i. e., u_{xxx}) with the highest order contri-

butions from the nonlinear terms (i. e., $u^2 u_x$), we get $N = 1$, so that

$$u(x, t) = \frac{u_0(x, t) + u_1(x, t)\phi(x, t)}{\phi(x, t)}. \quad (6)$$

We will stay with the general assumption that $\phi_x \neq 0$. When substituting the above expressions into (1) with symbolic computation, we make the coefficients of like powers of ϕ to vanish, so as to get the set of Painlevé-Bäcklund (PB) equations

$$\phi^{-4} : 6\gamma u_0^3 \phi_x - 6\beta(t)u_0 \phi_x^3 = 0, \quad (7)$$

$$\phi^{-3} : -6\gamma u_0^2 u_{0,x} + 12\gamma u_0^2 u_1 \phi_x + 6\beta(t)\phi_x^2 u_{0,x} \quad (8)$$

$$+ 6\beta(t)u_0 \phi_x \phi_{xx} + 6\alpha(t)u_0^2 \phi_x = 0,$$

$$\phi^{-2} : -6\gamma u_0^2 u_{1,x} - 12\gamma u_0 u_1 u_{0,x} + 6\gamma u_1^2 u_0 \phi_x \quad (9)$$

$$- 3\beta(t)\phi_x u_{0,xx} - 3\beta(t)\phi_{xx} u_{0,x} - \beta(t)u_0 \phi_{xxx}$$

$$- 6\alpha(t)u_0 u_{0,x} + 6\alpha(t)u_1 u_0 \phi_x - u_0 \phi_t = 0,$$

$$\phi^{-1} : -12\gamma u_0 u_1 u_{1,x} - 6\gamma u_1^2 u_{0,x} + \beta(t)u_{0,xxx} \quad (10)$$

$$- 6\alpha(t)u_0 u_{1,x} - 6\alpha(t)u_1 u_{0,x} + u_{0,t} = 0,$$

$$\phi^0 : u_1 \text{ needs to satisfy the original equation, i. e.,}$$

$$u_{1,t} - 6\alpha(t)u_1 u_{1,x} + \beta(t)u_{1,xxx} - 6\gamma u_1^2 u_{1,x} = 0. \quad (11)$$

The set of equations (6 - 11) constitutes an auto-Bäcklund transformation, if the set is consistent, i. e., if the set is solvable with respect to ϕ , $u_0(x, t)$, and $u_1(x, t)$ [9, 10]. Equation (7) brings out three solutions:

$$u_0(x, t) = \pm \sqrt{\frac{\beta(t)}{\gamma}} \phi_x \text{ or } u_0(x, t) = 0. \quad (12)$$

After substituting the non-trivial solutions into (8), we obtain

$$u_1(x, t) = \mp \frac{1}{2} \sqrt{\frac{\beta(t)}{\gamma}} \phi_{xx} \phi_x^{-1} - \frac{1}{2} \frac{\alpha(t)}{\gamma}. \quad (13)$$

Thus, we are able to find a family of exact analytic

solutions to (1) as

$$u(x, t) = \pm \sqrt{\frac{\beta(t)}{\gamma}} \phi_x \phi^{-1} \mp \frac{1}{2} \sqrt{\frac{\beta(t)}{\gamma}} \phi_{xx} \phi_x^{-1} - \frac{1}{2} \frac{\alpha(t)}{\gamma} \quad (14)$$

with the constraint equations (9 - 11) in terms of ϕ , ϕ_x , ϕ_{xx} , ϕ_{xxx} , $\alpha(t)$, $\beta(t)$, and γ .

We note that, once a Bäcklund transformation discovered and a set of “seed” solutions is given, one will be able to find an infinite number of solutions by repeated application of the transformation, i.e., to generate a hierarchy of solutions with increasing complexity. In the rest of the paper we will find a family of some exact analytic solutions to (1).

1. Sample Solution

A trial solution

$$\phi(x, t) = F(t) + G(t)e^{[A(t)x+B(t)]}, \quad (15)$$

where $F(t)$, $G(t)$, $A(t)$, and $B(t)$ are arbitrary real functions, is substituted into the constraint equation (9). Firstly, let us look for the trial solution corresponding to the positive $u_0(x, t)$, i.e., $u_0(x, t) = +\sqrt{\frac{\beta(t)}{\gamma}} \phi_x$. Equating to zero the coefficient of power of x yields the condition for $A(t)$

$$x : e^{[A(t)x+B(t)]} \sqrt{\beta(t)} G(t) A(t) \frac{dA(t)}{dt} = 0 \quad (16)$$

$$\Rightarrow A(t) = A = \text{nonzero constant.}$$

With this condition, we can then recast (10) and (11) as integrable differential equations

$$-A^3 \gamma \beta(t) + 3 A \alpha(t)^2 + \frac{d\beta(t)}{dt} \gamma + 2 \gamma \frac{d \ln G(t)}{dt} + 2 \gamma \frac{dB(t)}{dt} = 0 \quad (17)$$

and

$$A \frac{d\beta(t)}{dt} + 2 \frac{d\alpha(t)}{dt} \sqrt{\frac{\beta(t)}{\gamma}} = 0, \quad (18)$$

respectively. By solving (17) for $G(t)$, we get

$$G(t) = C_1 e^{\int^t \left[\frac{1}{2} \beta(\tau) A^3 - \frac{3}{2} \frac{A \alpha(\tau)^2}{\gamma} - \frac{1}{2\beta(\tau)} \frac{d\beta(\tau)}{d\tau} + \frac{dB(\tau)}{d\tau} \right] d\tau}, \quad (19)$$

where C_1 is an arbitrary constant. Finally, we obtain the constraint equation for $\alpha(t)$, $\beta(t)$, and γ from (18):

$$\beta(t) = \frac{(C_2 \gamma A + \sqrt{\gamma} \alpha(t))^2}{\gamma^2 A^2} \quad (20)$$

with an integration constant C_2 . Thus, we get the trial solution

$$\phi(x, t) = F(t) + \quad (21)$$

$$C_1 e^{\int^t \left[\frac{1}{2} \beta(\tau) A^3 - \frac{3}{2} \frac{A \alpha(\tau)^2}{\gamma} - \frac{1}{2\beta(\tau)} \frac{d\beta(\tau)}{d\tau} + \frac{dB(\tau)}{d\tau} \right] d\tau} \cdot e^{[Ax+B(t)]}$$

and find a class of solitary-wave solutions for (1)

$$u_+(x, t) = \frac{1}{A} \sqrt{\frac{\beta(t)}{\gamma}} \left[\frac{F(t)}{G(t)} \cdot e^{-(Ax+B(t))} + 1 \right] - \frac{A}{2} \sqrt{\frac{\beta(t)}{\gamma}} - \frac{\alpha(t)}{2\gamma} \quad (22)$$

with the constraint equation (20).

The other class of solutions, corresponding to $u_0(x, t) = -\sqrt{\frac{\beta(t)}{\gamma}} \phi_x$, is obtained by similar calculations as

$$u_-(x, t) = -\frac{1}{A(t)} \sqrt{\frac{\beta(t)}{\gamma}} \left[\frac{F(t)}{G} \cdot e^{-(A(t)x+B(t))} + 1 \right] + \frac{A(t)}{2} \sqrt{\frac{\beta(t)}{\gamma}} - \frac{\alpha(t)}{2\gamma}, \quad (23)$$

where physically meaningful terms, $A(t)$ and $B(t)$, are

$$A(t) = \frac{\alpha(t) + C_1 \sqrt{\gamma}}{\sqrt{\beta(t)} \gamma}, \quad (24)$$

$$B(t) = -\frac{1}{2} \int \frac{\mathcal{R}(t)}{S(t)} + C_2, \quad (25)$$

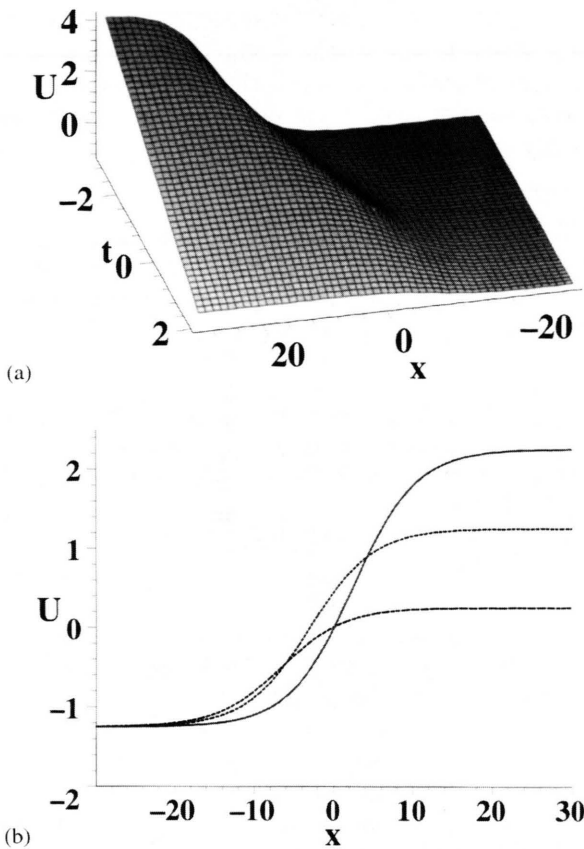


Fig. 1. (a) 3-dimensional plot of kink-type solitary-wave solutions for $u_+(x, t)$ with $\gamma = 1, C_1 = 10, C_2 = 4, A = 0.25, B(t) = t, \alpha(t) = t$, and $\beta(t) = 16(t+1)^2$; (b) The wave profiles for $u_+(x, t_i)$ for three different steps: $t_i = -1, 0, 1$ (solid, dash, dot-dash), respectively.

$$\begin{aligned} \mathcal{R}(t) \equiv & -2\gamma^2 \sqrt{\beta(t)} G(t) \alpha(t)' - 2\gamma^2 \sqrt{\beta(t)} G(t)' \alpha(t) \\ & + \gamma^{5/2} G(t) C_1^4 - 2\sqrt{\gamma} G(t) \alpha(t)^4 \\ & - 2\gamma^{5/2} \sqrt{\beta(t)} G(t)' C_1 - 2\gamma G(t) \alpha(t)^3 C_1 \\ & + 3\gamma^{3/2} G(t) C_1^2 \alpha(t)^2 + 4\alpha(t) G(t) C_1^3 \gamma^2, \end{aligned}$$

$$S(t) \equiv \sqrt{\beta(t)} G(t) \gamma^2 (\alpha(t) + C_1 \sqrt{\gamma}),$$

and the constraint equation is

$$\beta(t) = C_2 \left(\gamma^{5/2} C_1^2 + 2\gamma^2 \alpha(t) C_1 + \gamma^{3/2} \alpha(t)^2 \right). \quad (26)$$

In the following, to explicitly demonstrate the solitary-wave solutions, we plot (22) and (23) by choosing

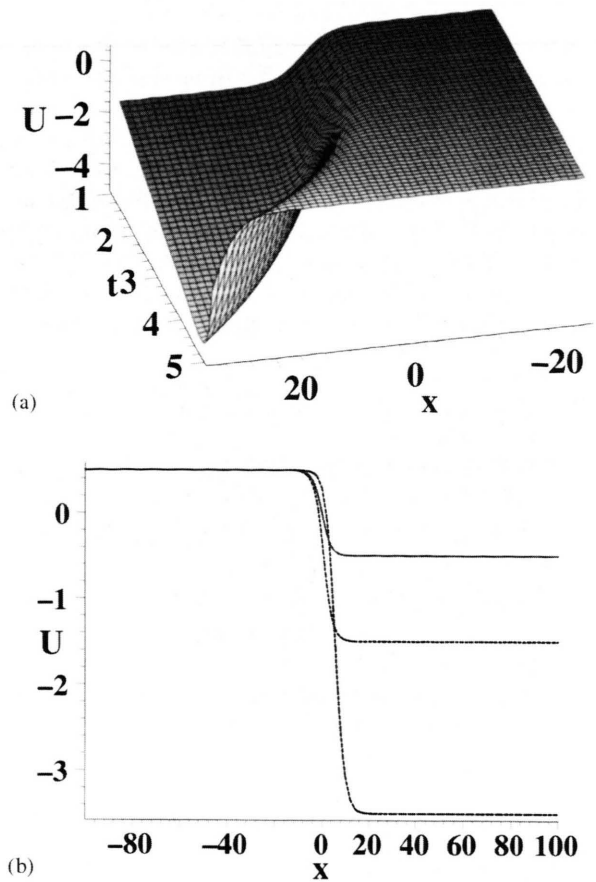


Fig. 2. (a) 3-dimensional plot of anti-kink-type solitary-wave solutions for $u_-(x, t)$ with $\gamma = 1, C_1 = 1, C_2 = 4, F(t) = t, G = 1, \alpha(t) = t$, and $\beta(t) = 4(t+1)^2$; (b) The wave profiles for $u_-(x, t_i)$ for three different steps: $t_i = 0, 1, 3$ (solid, dash, dot-dash), respectively.

some proper model coefficients for $\alpha(t)$, γ , and $\beta(t)$ which satisfy the above constraints. As for a physically interesting case, we choose $\alpha(t) = t$ for which the nonlinear coefficient grows as the wave propagates and study how the wave profile may change. We set $\gamma = 1, C_1 = 10, C_2 = 4, A = 0.25, B(t) = t$, and $\beta(t) = 16(t+1)^2$ (required by the constraint (20)) for $u_+(x, t)$; $\gamma = 1, C_1 = 1, C_2 = 4, F(t) = t, G = 1$, and $\beta(t) = 4(t+1)^2$ (required by the constraint (26)) for $u_-(x, t)$ with $A(t)$ and $B(t)$ in terms of the chosen parameters. In Figs. 1a and 1b we show a 3-dimensional plot of $u_+(x, t)$ and the kink-type solitary-wave profile of $u_+(x, t_i)$ for three different steps: $t_i = -1, 0, 1$ (solid, dash, dot-dash), respectively. We note that the amplitude of the solitary wave decreases with increasing t . This fact is evident from (22) and noting that the

dispersion coefficient grows $\sim t^2$. On the other hand, the 3-dimensional plot of $u_-(x, t)$ and the anti-kink-type solitary-wave profiles for three different steps $t_i = 0, 1, 3$ (solid, dash, dot-dash), respectively, are plotted in Figs. 2a and 2b. Similar to the positive solution, the anti-kink-type wave profile decreases with increasing t , which originates from the non-trivial dependences of the physically meaningful terms $A(t)$ and $B(t)$ on t as shown in (24) and (25).

To sum up, with symbolic calculations and the truncated Painlevé expansion analysis, we showed that

Bäcklund transformations exist for the modified KdV equation with t -dependent coefficients. We found two classes of analytic kink and anti-kink type solitary-wave solutions which are different from the previously obtained analytic solutions in [3, 5, 6].

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